

Gauge Supergravities for All Odd Dimensions

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Recently proposed supergravity theories in odd dimensions whose fields are connection one-forms for the minimal supersymmetric extensions of anti-de Sitter gravity are discussed. Two essential ingredients are required for this construction: (1) The superalgebras, which extend the adS algebra for different dimensions, and (2) the Lagrangians, which are Chern–Simons $(2n - 1)$ -forms. The first item completes the analysis of van Holten and Van Proeyen, which was valid for $N = 1$ only. The second ensures that the actions are invariant by construction under the gauge supergroup and, in particular, under local supersymmetry. Thus, unlike standard supergravity, the local supersymmetry algebra closes off-shell and without requiring auxiliary fields. The superalgebras are constructed for all dimensions and they fall into three families: $osp(m|N)$ for $D = 2, 3, 4, \text{ mod } 8$, $osp(N|m)$ for $D = 6, 7, 8, \text{ mod } 8$, and $su(m - 2, 2|N)$ for $D = 5 \text{ mod } 4$, with $m = 2^{\lfloor D/2 \rfloor}$. The Lagrangian is constructed for $D = 5, 7$, and 11 . In all cases the field content includes the vielbein (e_{μ}^a), the spin connection (ω_{μ}^{ab}), N gravitini (Ψ_{μ}^i), and some extra bosonic “matter” fields which vary from one dimension to another.

1. INTRODUCTION

This is an expanded version of a recent paper [1] and lecture [2], where most of the preliminary results were announced.

Three of the four fundamental forces of nature are consistently described by Yang–Mills (YM) quantum theories. Gravity, the fourth fundamental interaction, resists quantization in spite of several decades of intensive research in this direction. This is intriguing in view of the fact that general relativity (GR) and YM theories have a deep geometrical nature based on the gauge principle. Why do two theories constructed on almost the same mathematical foundation produce such radically different physical behaviors? What is the obstruction to the application of the

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methods of YM quantum field theory to gravity? The final answer to these questions is beyond the scope of this paper; however, one can note a difference between YM and GR which might turn out to be an important clue: YM theory is defined on a fiber bundle, with the connection as the dynamical object, whereas the dynamical fields of GR cannot be interpreted as components of a connection.

The closest one could get to a connection formulation for GR is the Palatini formalism, with the Hilbert action

$$I[\omega, e] = \int \epsilon_{abcd} R^{ab} \wedge e^a \wedge e^b \quad (1)$$

where $R^{ab} = d\omega^{ab} + \omega_c^a \wedge \omega_b^c$ is the curvature two-form. This action is sometimes—mistakenly—claimed to describe a gauge theory for local translations. If ω and e were the components of the Poincaré connection associated to local translations, they should transform as

$$\delta\omega^{ab} = 0, \quad \delta e^a = D\lambda^a = d\lambda^a + \omega_b^a \wedge \lambda^b \quad (2)$$

Invariance of (1) under (2) would require the torsion-free condition

$$T^a = de^a + \omega_b^a \wedge e^b = 0 \quad (3)$$

which is an equation of motion for the action (1). This means that the invariance of the action (1) under (4) does not result from the transformation properties of the fields alone, but is a property of their dynamics as well.

The error stems from the identification between local translations in the base manifold (diffeomorphisms)

$$x^\mu \rightarrow x'^\mu = x^\mu + \zeta^\mu(x) \quad (4)$$

which is a genuine invariance of the action (1), and local translations in the tangent space (2).

The torsion-free condition, being one of the field equations, implies that local translational invariance is at best an *on-shell* symmetry, which would probably not survive quantization.

Since the invariance of the Hilbert action under general coordinate transformations (4) is reflected in the closure of the first-class Hamiltonian constraints in the Dirac formalism, one could try to push the analogy between the Hamiltonian constraints H_μ and the generators of a gauge algebra. However, the fact that the constraint algebra requires structure *functions*, which

depend on the dynamical fields, is another indication that the generators of diffeomorphism invariance of the theory do not form a Lie algebra, but an open algebra (see, e.g., ref. 3).

More precisely, the subalgebra of spatial diffeomorphisms *is* a genuine Lie algebra in the sense that its structure constants are independent of the dynamical fields of gravitation,

$$[H_i, H'_j] \sim H'_j \delta_{li} - H'_i \delta_{lj} \quad (5)$$

whereas the generators of timelike diffeomorphisms are the offending ones: they form an open algebra,

$$[H_\perp, H'_\perp] \sim g^{ij} H'_j \delta_{li} \quad (6)$$

This comment is particularly appropriate in a CS theory, where spatial diffeomorphisms are always part of the true gauge symmetries of the theory. The generators of timelike displacements (H_\perp), on the other hand, are combinations of the internal gauge generators and the generators of spatial diffeomorphism, and therefore do not generate independent symmetries [4].

2. SUPERGRAVITY

For some time it was hoped that the nonrenormalizability of GR could be cured by supersymmetry. However, the initial glamor of supergravity (SUGRA) as a mechanism for taming the wild ultraviolet divergences of pure gravity was eventually spoiled by the realization that it, too, would lead to a nonrenormalizable answer. Again, one can see that SUGRA is not a gauge theory in the sense of a fiber bundle and that the local symmetry algebra closes naturally only on shell. The algebra can be made to close off shell at the cost of introducing auxiliary fields, but they are not guaranteed to exist for all D and N [5].

Whether the lack of fiber bundle structure is the ultimate reason for the nonrenormalizability of gravity remains to be proven. However, it is certainly true that if GR could be formulated as a gauge theory, the chances of proving its renormalizability would clearly grow. In three spacetime dimensions both GR and SUGRA define renormalizable quantum theories. It is strongly suggestive that precisely in this case both theories can also be formulated as gauge theories on a fiber bundle [6]. It might seem that the exact solvability miracle was due to the absence of propagating degrees of freedom in three-dimensional gravity, but the key ingredient of the miracle can be traced to the fiber bundle structure of the Chern–Simons (CS) form of those systems.

There are other known examples of gravitation theories in odd dimensions which are genuine (off-shell) gauge theories for the anti-de Sitter (adS)

or Poincaré groups [7–10]. These theories, as well as their supersymmetric extensions, have propagating degrees of freedom [4] and are CS systems for the corresponding groups as shown in ref. 11.

2.1. From Rigid Supersymmetry to Supergravity

Rigid SUSY can be understood as an extension of the Poincaré algebra by including supercharges which are the “square roots” of the generators of rigid translations, $\{\bar{Q}, Q\} \sim \Gamma \cdot P$. Roughly speaking, the traditional strategy to generalize this idea to local SUSY was to substitute the momentum $P_\mu = i\partial_\mu$ by the generators of diffeomorphisms, $\{\bar{Q}, Q\} \sim \Gamma \cdot \mathcal{H}$. The resulting theory has an on-shell local supersymmetry algebra.

An alternative point of view—which is the one we advocate here—is to construct the supersymmetry on the tangent space and not on the base manifold. This approach is more natural if one recalls that spinors provide a basis of irreducible representations for $SO(N)$, and not for $GL(N)$. Thus, spinors are naturally defined relative to a local frame on the tangent space rather than in the coordinate basis. This idea has been successfully applied by Chamseddine in five dimensions [8], and by us for pure gravity [9, 10] and in supergravity [1, 11]. The basic point is to replicate the 2 + 1 “miracle” in higher dimensions. The construction has been carried out for spacetimes whose tangent space has adS symmetry in ref. 1, and for its Poincaré contraction in ref. 11.

In ref. 11 a family of theories in odd dimensions, invariant under the supertranslation algebra whose bosonic sector contains the Poincaré generators, was presented. The anticommutator of the supersymmetry generators gives a translation plus a tensorial “central” extension,

$$\{Q^\alpha, \bar{Q}_\beta\} = -i(\Gamma^\alpha)_\beta^\alpha P_\alpha - i(\Gamma^{abcde})_\beta^\alpha Z_{abcde} \quad (7)$$

The commutators of Q , \bar{Q} , and Z with the Lorentz generators can be read off from their tensorial character. All the remaining commutators vanish. This algebra is the continuation to all odd-dimensional spacetimes of the $D = 10$ superalgebra of ref. 12 and yields supersymmetric theories with off-shell Poincaré superalgebra. The existence of these theories suggests that there should be similar supergravities based on the adS algebra.

2.2. Assumptions of Standard Supergravity

Three implicit assumptions are usually made in the construction of standard SUGRA:

(i) The fermionic and bosonic fields in the Lagrangian should come in combinations such that their propagating degrees of freedom are equal in

number. This is usually achieved by adding to the graviton and the gravitini a number of lower spin fields ($s < 3/2$) [13]. This matching, however, is not necessarily true in adS space, nor in Minkowski space if a different representation of the Poincaré group (e.g., the adjoint representation) is used [14].

The other two assumptions concern the purely gravitational sector. They are as old as general relativity itself and are dictated by economy: (ii) Gravitons are described by the Hilbert action (plus a possible cosmological constant), and (iii) the spin connection and the vielbein are not independent fields, but are related through the torsion equation. The fact that the supergravity generators do not form a closed off-shell algebra can be traced back to these assumptions.

The procedure behind (i) is tightly linked to the idea that the fields should be in a *vector* representation of the Poincaré group [14] and that the kinetic terms and couplings are such that the counting of degrees of freedom works as in a minimally coupled gauge theory. This assumption comes from the interpretation of supersymmetric states as represented by the in and out plane waves in an asymptotically free, weakly interacting theory in a Minkowskian background. These conditions are not necessarily met for a CS theory in an asymptotically adS background. Apart from the difference in background, which requires a careful treatment of the unitary irreducible representations of the asymptotic symmetries [15], the counting of degrees of freedom in CS theories is completely different from the one for the same connection one-forms in a YM theory.

2.3. Lanczos–Lovelock Gravity

For $D > 4$, assumption (ii) is an unnecessary restriction in the available theories of gravitation. In fact, the most general action for gravity—generally covariant and with second-order field equations for the metric—is a polynomial of degree $[D/2]$ in the curvature, first discussed by Lanczos [16] for $D = 5$ and, in general, by Lovelock [17, 18]. This action contains the same degrees of freedom as the Hilbert action [19] and is the most general low-energy effective theory of gravity derived from string theory [20].

The Lanczos–Lovelock (LL) theory contains as a particular case the Einstein–Hilbert (EH) theory, but they are in general dynamically quite different. The classical solutions of the LL theory are not perturbatively related to those of Einstein’s theory. For instance, it was observed that the time evolution of the classical solutions in the LL theory starting from a generic initial state can be unpredictable, whereas the EH theory defines a well-posed Cauchy problem [19, 21]. Moreover, the LL theory

has a large number of dimensional constants in contrast with the two constants of the EH action (g and Λ) [22, 23, 9]). This last feature would seem to indicate that renormalizability would be even more remote for the LL theory than in ordinary gravity.

However, there is an exceptional case when all the constants of the LL action are related to each other in a particular way (see next section). The resulting system exhibits a larger symmetry; the theory can be formulated in terms of a dimensionless connection 1-form and a unique dimensionless constant which is also quantized [10]. In this case, the LL theory becomes a CS system in which the spin connection and the vielbein are parts of an adS connection. A particular case of this occurs naturally in $2 + 1$ dimensions in the presence of a cosmological constant. This is the secret behind the integrability of gravity in $2 + 1$ dimensions as demonstrated in ref. 6.

2.4. Torsion

Assumption (iii) implies that torsion does not contain independently propagating degrees of freedom, and its equations are identities enforced by fiat on the fields. This is the essence of the metric approach to GR in which parallel transport and distance are not independent notions, but are related through the Christoffel symbols. There is no natural justification for this assumption and it was the leitmotif of the historic discussion between Einstein and Cartan [24].

In four dimensions if e and ω are varied independently in the action (1), the equation for ω implies $T^a = 0$ (in the absence of matter). This might seem sufficient to justify assumption (iii) since the equation for ω is algebraic and could in principle be used to express ω in terms of the remaining fields. However, it is not always possible to solve this algebraic equation for ω if $D > 4$.

Another serious consequence of the torsion-free conditions is that the dynamical dependence between ω and e introduced by (3) spoils the possibility of interpreting the local translational invariance as a gauge symmetry of the action. Indeed, taking the condition $T^a = 0$ as a definition would imply

$$\delta\omega^{ab} = \frac{\delta\omega^{ab}}{\delta e^c} \delta e^c \neq 0 \quad (8)$$

in contrast with (2). Thus, the spin connection and the vielbein cannot be identified as the compensating fields for local Lorentz rotations and translations, respectively, as can be done in $D = 3$. Thus, general relativity in $D = 2n \geq 4$ cannot be formulated purely as a gauge theory on a fiber bundle.

In our construction [1, 11] ω and e are assumed to be dynamically independent and thus torsion necessarily contains propagating degrees of freedom, represented by the contorsion tensor $k_{\mu}^{ab} := \omega_{\mu}^{ab} - \bar{\omega}_{\mu}^{ab}(e, \dots)$, where $\bar{\omega}$ is the spin connection which solves the classical torsion equation in terms of the remaining fields.

3. GAUGE GRAVITY AND GAUGE SUPERGRAVITY

3.1 Chern–Simons Gravity in $2n - 1$ Dimensions

For $D = 2n - 1$, the particular choice of the LL Lagrangian reads

$$L_{G,2n-1}^{adS} = \sum_{p=0}^{n-1} \alpha_p L^p \tag{9}$$

where

$$L_G^p = \epsilon_{a_1 \dots a_D} R^{a_1 a_2} \dots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \dots e^{a_D} \tag{10}$$

Here the wedge product is understood and the subscript G stands for “torsion-free gravity.” We note that although torsion does not explicitly enter in (10), for $D \geq 5$ this system has propagating torsion.

If one chooses $\alpha_p = \kappa(D - 2p)^{-1} \binom{n-1}{p} l^{2p-D}$, the invariance of (9) under the Lorentz group extends to the adS group. The constant l has dimensions of length and its purpose is to render the action dimensionless, allowing the interpretation of ω and e as components of the adS connection [23];

$$W^{AB} = \begin{bmatrix} \omega^{ab} & e^a l \\ -e^b l & 0 \end{bmatrix} \tag{11}$$

where $A, B = 1, \dots, D + 1$. The resulting Lagrangian is an adS-CS form in the sense that its exterior derivative is the Euler form in $2n$ dimensions [8, 9, 23],

$$dL_{G,2n-1}^{adS} = \kappa \epsilon_{A_1 \dots A_{2n}} R^{A_1 A_2} \dots R^{A_{2n-1} A_{2n}} \tag{12}$$

where R^{AB} is the adS curvature and κ is quantized [10] (in the following we will set $\kappa = l = 1$).

Apart from the Euler-CS form discussed above, there are the standard CS $(2p - 1)$ -forms L_{2p-1}^* related to the p th Chern character for the Lorentz connection,

$$dL_{2p-1}^*(\omega_b^a) = c_p \tag{13}$$

where $c_p =: \text{Tr}[(R_b^a)^p]$.

In general, a Chern–Simons D-form is defined by the condition that its exterior derivative be an invariant homogeneous polynomial of degree n in the curvature, that is, a characteristic class. In the examples above, (12) is the CS form for the Euler class $2n$ -form, while the exotic Lagrangians are related to different combinations of Chern characters (see Appendix A).

Thus, a generic CS action in $2n - 1$ dimensions for a Lie algebra g can be written as

$$dL_{2n-1}^g = \langle \mathbf{F}^n \rangle \tag{14}$$

where $\langle \cdot \rangle$ stands for a multilinear function in the Lie algebra g , invariant under cyclic permutations such as Tr , for an ordinary Lie algebra, or STr , in the case of a superalgebra. For $D = 3$ the “exotic gravity” Lagrangian

$$L_{T,3}^{adS} = \omega_a^a d\omega_a^b + \frac{2}{3} \omega_b^a \omega_c^b \omega_a^c + 2e_a T^a \tag{15}$$

is a CS form for the adS group $SO(4)^2$, whose exterior derivative is the Pontryagin form in four dimensions [Second Chern character for $SO(4)$][6],

$$dL_{T,3}^{adS} = R_B^A R_A^B \tag{16}$$

where R_B^A is the Riemann curvature 2-form in four dimensions.

Similar exotic actions associated to the Chern characters in $4k$ dimensions exist in $D = 4k - 1$ [25]. The gravitational Chern characters vanish for $p = 2n + 1$ (the trace of product of an odd number of Riemann curvatures vanishes) [26], hence there are no adS CS Lagrangians in $D = 4k + 1$. For $D = 4k - 1$, the number of possible exotic forms grows as the partitions of k , in correspondence with the number of composite Chern invariants of the form $\prod_i c_{p_i}$, with $\sum_i p_i = 4k$. Out of all these forms, we will be interested in particular combinations of them which, in the spinorial representation of $SO(4k)$, can be written as [1]

$$dL_{T,4k-1}^{adS} = - \text{Tr} \left[\left(\frac{1}{4} R^{AB} \Gamma_{AB} \right)^{2k} \right] \tag{17}$$

It is important to note that in this Lagrangian, as well as in (16), torsion appears explicitly. For example, in seven dimensions one finds

$$\begin{aligned} L_{T,7}^{adS} &= L_7^*(\omega) - \frac{3}{4} (R_b^a R_a^b + 2[T^a T_a - R^{ab} e_a e^b]) L_3^*(\omega) \\ &\quad - \left(\frac{3}{2} R_b^a R_a^b + 2T^a T_a - 4R^{ab} e_a e_b \right) T^a e_a + 4T_a R_b^a R_a^b e^c \end{aligned}$$

²For simplicity we will not distinguish between different signatures. Thus, the adS algebra in D dimensions will be denoted as $so(D + 1)$.

In eight dimensions there are four topological invariants [27], which give rise to four corresponding CS forms in seven dimensions. $L_{T,7}^{adS}$ is a particular linear combination of those CS forms. For further discussion on these invariants, see refs. 25 and 27.

If one allows the presence of torsion explicitly, the maximal extension of the LL Lagrangian is found. This is the most general D -form invariant under local Lorentz rotations constructed out of the vielbein, the spin connection, and their derivatives (without using the metric) [26]. The Lagrangian (17) is a particular representative of this family in which the coefficients are chosen so as to make the action locally invariant under the adS group $SO(D + 1)$ [25].

The exterior derivative of a possible CS Lagrangian for a given group in $D = 2n - 1$ dimensions has the form $g_{a_1 \dots a_n} F^{a_1} \dots F^{a_n}$, where $g_{a_1 \dots a_n}$ is an invariant tensor of the algebra. Thus, the problem of finding all possible CS Lagrangians is equivalent to finding all possible invariant tensors of rank n in the Lie algebra. This is in general an open problem, related to the number of Casimir invariants for a given Lie group. For the groups relevant for supergravity discussed below, like $OSp(32|1)$, the number of invariant tensors can be rather large. Most of these invariants would give rise to bizarre Lagrangians and the real problem is to find the appropriate invariants that describe sensible theories.

The R.H.S. of (17) is a particular form of (14) where $\langle \cdot \rangle$ is the ordinary trace over spinor indices. Other possibilities of the form $\langle \mathbf{F}^{n-p} \rangle \langle \mathbf{F}^p \rangle$ are not used in our construction as they would not lead to the minimal supersymmetric extensions of adS containing the Hilbert action. In the supergravity theories discussed below, the gravitational sector is given by

$$\pm \frac{1}{2^n} L_{G,2n-1}^{adS} - \frac{1}{2} L_{T,2n-1}^{adS}$$

The \pm sign corresponds to the two choices of inequivalent representations of Γ 's, which in turn reflect the two chiral representations in $D + 1$. As in the three-dimensional case, the supersymmetric extensions of L_G or any of the exotic Lagrangians such as L_T require using both chiralities, thus doubling the algebras. Here we choose the + sign, which gives the minimal superextension [25].

The bosonic theory outlined above is our starting point. The idea now is to construct its supersymmetric extension. A possible approach would be to study the superalgebras containing anti-de Sitter as a subalgebra, define a connection—in the adjoint representation—and construct the CS form with it. This construction must clearly give the right result, but it would be rather difficult to proceed formally without an explicit representation. Knowing this, we take a less formal path, expressing the adjoint representation in terms of the Dirac matrices of the appropriate dimension. This is not a wild guess

because the generators of the Dirac algebra $\{I, \Gamma^a, \Gamma^{ab}, \dots\}$ constituting a basis for all square matrices. The advantage of this approach is that it provides an explicit representation of the algebra and writing the Lagrangians is straightforward.

3.2 Gauge Supergravity

The supersymmetric extensions of the adS algebras in $D = 2, 3, 4$, mod 8, were studied by van Holten and Van Proeyen [12]. They added one Majorana supersymmetry generator to the adS algebra and found all the $N = 1$ extensions demanding closure of the full superalgebra. In spite of the fact that the algebra for $N = 1$ adS supergravity in 11 dimensions was conjectured in 1978 to be $osp(32|1)$ by Cremer *et al.* [28], and this was confirmed in [12], nobody constructed a supergravity action for this algebra in the intervening 20 years. One reason for the lack of interest in the problem might have been the fact that the $osp(32|1)$ algebra contains generators which are Lorentz tensors of rank higher than two.

Apart from the assumptions mentioned above, supergravity algebras were traditionally limited to generators which are Lorentz tensors up to second rank. This constraint was based on the observation that elementary particle states of spin higher than two would be inconsistent [29]. However, this does not rule out the relevance of those tensor generators in theories of extended objects [30]. In fact, it is quite common nowadays to find algebras like the M -brane superalgebra [31, 32],

$$\{Q, \tilde{Q}\} \sim \Gamma^a P_a + \Gamma^{ab} Z_{ab} + \Gamma^{abcde} Z_{abcde} \quad (18)$$

4. SUPERALGEBRA AND CONNECTION

The smallest superalgebra containing the adS algebra in the bosonic sector is found following the same approach as in ref. 12, but lifting the restriction of $N = 1$ [25]. The result, for odd $D > 3$, is given in Table I (see Appendix B).

Table I

D	S-Algebra	Conjugation matrix	Internal metric
$8k - 1$	$osp(N m)$	$C^T = C$	$u^T = -u$
$8k + 3$	$osp(m N)$	$C^T = -C$	$u^T = u$
$4k + 1$	$su(m N)$	$C^\dagger = C$	$u^\dagger = u$

In each of these cases, $m = 2^{[D/2]}$ and the connection takes the form

$$\mathbf{A} = \frac{1}{2} \omega^{ab} J_{ab} + e^a J_a + \frac{1}{r!} b^{[r]} Z_{[r]} + \frac{1}{2} (\bar{\Psi}^i Q_i - \bar{Q}^j \psi_j) + \frac{1}{2} a_{ij} M^{ij} \tag{19}$$

The generators J_{ab}, J_a span the adS algebra, Q_α^i generate (extended) supersymmetry transformations, and $[r]$ denotes a set of r antisymmetrized Lorentz indices. The Q 's transform as vectors under the action of M_{ij} and as spinors under the Lorentz group. Finally, the Z 's complete the extension of adS into the larger algebras $so(m), sp(m)$, or $su(m)$.

In (19) $\psi^i = \psi_j^\dagger C u^{ji} (\bar{\psi}^i = \bar{\psi}_j^\dagger C u^{ji})$ for $D = 4k + 1$, where C and u are given in Table I. These algebras admit $(m + N) \times (m + N)$ matrix representations [48], where the J and Z have entries in the $m \times m$ block, the M_{ij} in the $N \times N$ block, while the fermionic generators Q have entries in the complementary off-diagonal blocks.

Under a gauge transformation, \mathbf{A} transforms by $\delta \mathbf{A} = \nabla \lambda$, where ∇ is the covariant derivative for the same connection \mathbf{A} . In particular, under a supersymmetry transformation, $\lambda = \bar{\epsilon}^i Q_i - \bar{Q}^i \epsilon_i$, and

$$\delta_\epsilon \mathbf{A} = \begin{bmatrix} \epsilon^k \bar{\psi}_k - \bar{\psi}^k \epsilon_k & D \epsilon_j - \bar{\psi}^i \epsilon_j \\ -D \bar{\epsilon}^i & \bar{\epsilon}^i \psi_j - \bar{\psi}^i \epsilon_j \end{bmatrix} \tag{20}$$

where D is the covariant derivative on the bosonic connection,

$$D \epsilon_j = \left(d + \frac{1}{2} \left[e^a \Gamma_a + \frac{1}{2} \omega^{ab} \Gamma_{ab} + \frac{1}{r!} b^{[r]} \Gamma_{[r]} \right] \right) \epsilon_j - a_j^i \epsilon_i.$$

4.1. $D=5$ Supergravity

In this case, as in every dimension $D = 4k + 1$, there is no torsional Lagrangian L_T due to the vanishing of the Pontryagin $4k + 2$ -forms for the Riemann curvature. This fact implies that the local supersymmetric extension will be of the form $L = L_G + \dots$

As shown in Table I, the appropriate *adS* superalgebra in five dimensions is $su(2, 2|N)$, whose generators are $K, J_a, J_{ab}, Q^\alpha, \bar{Q}_\beta, M^{ij}$, with $a, b = 1, \dots, 5$ and $i, j = 1, \dots, N$. The connection is $\mathbf{A} = bK + e^a J_a + \frac{1}{2} \omega^{ab} J_{ab} + a_{ij} M^{ij} + \psi^i Q_i - \bar{Q}^j \psi_j$, so that in the adjoint representation

$$\mathbf{A} = \begin{bmatrix} \Omega_\beta^\alpha & \psi_j^\alpha \\ -\bar{\psi}_\beta^i & A_j^i \end{bmatrix} \tag{21}$$

with $\Omega_\beta^\alpha = 1/2 (\frac{i}{2} b I + e^a \Gamma_a + \omega^{ab} \Gamma_{ab})_\beta^\alpha$, $A_j^i = (iN) \delta_j^i b + a_j^i$, and $\bar{\psi}_\beta^i = \psi^{\dagger \alpha j} G_{\alpha\beta}$. Here G is the Dirac conjugate (e.g., $G = {}_i \Gamma^0$). The curvature is

$$\mathbf{F} = \begin{bmatrix} \bar{R}_\beta^\alpha & D\psi_j^\alpha \\ -D\bar{\psi}_\beta^i & \bar{F}_j^i \end{bmatrix} \tag{22}$$

where

$$\begin{aligned} D\psi_j^\alpha &= d\psi_j^\alpha + \Omega_\beta^\alpha \psi_j^\beta - A_j^i \psi_i^\alpha \\ \bar{R}_\beta^\alpha &= R_\beta^\alpha - \psi_i^\alpha \bar{\psi}_\beta^i \\ \bar{F}_j^i &= F_j^i - \bar{\psi}_\beta^i \psi_j^\beta \end{aligned} \tag{23}$$

Here $F_j^i = dA_j^i + A_k^i A_j^k + (i/N)db\delta_j^i$ is the $su(N)$ curvature, and $R_\beta^\alpha = d\Omega_\beta^\alpha + \Omega_\sigma^\alpha \Omega_\beta^\sigma$ is the $u(2,2)$ curvature. In terms of the standard $(2n - 1)$ -dimensional fields, R_β^α can be written as

$$R_\beta^\alpha = \frac{i}{4} db \delta_\beta^\alpha + \frac{1}{2} [T^a \Gamma_a + (R^{ab} + e^a e^b) \Gamma_{ab}]_\beta^\alpha \tag{24}$$

In six dimensions the only invariant form is

$$P = iStr[\mathbf{F}^3] \tag{25}$$

which in this case reads

$$\begin{aligned} P &= Tr[R^3] - Tr[F^3] \\ &\quad + 3[D\bar{\psi}(\bar{R} + \bar{F})D\psi - \bar{\psi}(R^2 - F^2 + [R - F](\psi)^2)\psi] \end{aligned} \tag{26}$$

where $(\psi)^2 = \bar{\psi}\psi$. The resulting five-dimensional CS density can be decomposed as a sum of a gravitational part, a b -dependent piece, an $su(N)$ gauge part, and a fermionic term,

$$L = L_G^{adS} + L_b + L_{SU(N)} + L_F \tag{27}$$

with

$$\begin{aligned} L_G^{adS} &= \frac{1}{8} \epsilon_{abcde} \left(R^{ab} R^{cd} e^e + \frac{2}{3} R^{ab} e^c e^d e^e + \frac{1}{5} e^a e^b e^c e^d e^e \right) \\ L_b &= -\left(\frac{1}{N^2} - \frac{1}{4^2} \right) (db)^2 b - \frac{3}{2} \left(T^a T_a - R^{ab} e_a e_b - \frac{1}{2} R^{ab} R_{ab} \right) b + \frac{3}{N} b f_j^i f_j^i \\ L_{SU(N)} &= -\left(a_j^i da_k^j da_i^k + \frac{3}{2} a_j^i a_k^j a_l^k da_i^l + \frac{3}{5} a_j^i a_k^j a_l^k a_m^l a_m^i \right) \\ L_F &= 3[\bar{\psi}(\bar{R} + \bar{F})D\psi - (\psi)^2(\bar{\psi}D\psi)] \end{aligned} \tag{28}$$

The action is invariant under local gauge transformations, which contain the local SUSY transformations

$$\begin{aligned}
 \delta e^a &= -\frac{1}{2} (\bar{\epsilon}^i \Gamma^a \psi_i - \bar{\psi}^i \Gamma^a \epsilon_i) \\
 \delta \omega^{ab} &= \frac{1}{4} (\bar{\epsilon}^i \Gamma^{ab} \psi_j - \bar{\psi}^i \Gamma^{ab} \epsilon_j) \\
 \delta b &= i(\bar{\epsilon}^i \psi_i - \bar{\psi}^i \epsilon_i) \\
 \delta \psi_i &= D \epsilon_i \\
 \delta \bar{\psi}^i &= D \bar{\epsilon}^i \\
 \delta a_j^i &= i(\bar{\epsilon}^i \psi_j - \bar{\psi}^i \epsilon_j)
 \end{aligned} \tag{29}$$

As in 2 + 1 dimensions, the Poincaré supergravity theory is recovered contracting the super *adS* group. Consider the following rescaling of the fields:

$$\begin{aligned}
 e^a &\rightarrow \frac{1}{\alpha} e^a \\
 \omega^{ab} &\rightarrow \omega^{ab} \\
 b &\rightarrow \frac{1}{3\alpha} b \\
 \psi_i &\rightarrow \frac{1}{\sqrt{\alpha}} \psi_i \\
 \bar{\psi}^i &\rightarrow \frac{1}{\sqrt{\alpha}} \bar{\psi}^i \\
 a_j^i &\rightarrow a_j^i
 \end{aligned} \tag{30}$$

Then, if the gravitational constant is also rescaled as $\kappa \rightarrow \alpha\kappa$, in the limit $\alpha \rightarrow \infty$ the action becomes that in ref. 11, plus an *su(N)* CS form,

$$\begin{aligned}
 I &= \frac{1}{8} \int [\epsilon_{abcde} R^{ab} R^{cd} e^a - R^{ab} R_{abb} \\
 &\quad - 2R^{ab} (\bar{\psi}^i \Gamma_{ab} D \psi_i + D \bar{\psi}^i \Gamma_{ab} \psi_i) + L_{su(N)}]
 \end{aligned} \tag{31}$$

The rescaling (30) induces a contraction of the super *adS* algebra *su(mlN)* into [Super Poincaré] \otimes *su(N)*, where the second factor is an automorphism.

4.2. D = 7 Supergravity

The smallest adS superalgebra in seven dimensions is *osp(2|8)*. The connection (19) is $\mathbf{A} = \frac{1}{2} \omega^{ab} J_{ab} + e^a J_a + \bar{Q}^i \psi_i + \frac{1}{2} a_{ij} M^{ij}$, where M^{ij} are the generators of *sp(2)*. In the representation given above, the bracket $\langle \cdot \rangle$ is the

supertrace and, in terms of the component fields appearing in the connection, the CS form is

$$L_7^{osp(2|8)}(\mathbf{A}) = 2^{-4} L_{G,7}^{adS}(\omega, e) - \frac{1}{2} L_{T,7}^{adS}(\omega, e) - L_j^{*Sp(2)}(a) + L_F(\psi, \omega, e, a) \tag{32}$$

Here the fermionic Lagrangian is

$$L_F = 4\bar{\Psi}^j(R^2\delta_j^i + Rf_j^i + (f^2)_{jj}^i)D\Psi_i + 4(\bar{\Psi}^i\Psi_j)[(\bar{\Psi}^j\Psi_k)(\bar{\Psi}^k D\Psi_i) - \bar{\Psi}^j(R\delta_i^k + f_i^k)D\Psi_k] - 2(\bar{\Psi}^i D\Psi_j)[\bar{\Psi}^j(R\delta_i^k + f_j^k)\Psi_k + D\bar{\Psi}^j D\Psi_i]$$

where $f_j^i = da_j^i + a_k^i a_j^k$, and $R = \frac{1}{4}(R^{ab} + e^a e^b)\Gamma_{ab} + \frac{1}{2} T^a \Gamma_a$ are the $sp(2)$ and $so(8)$ curvatures, respectively. The supersymmetry transformations (20) read

$$\delta e^a = \frac{1}{2} \bar{\epsilon}^i \Gamma^a \psi_i, \quad \delta \omega^{ab} = -\frac{1}{2} \bar{\epsilon}^i \Gamma^{ab} \psi_i$$

$$\delta \psi_i = D\epsilon_i, \quad \delta a_j^i = \bar{\epsilon}^i \psi_j - \bar{\Psi}^i \epsilon_j$$

Standard seven-dimensional supergravity is an $N = 2$ theory (its maximal extension is $N = 4$), whose gravitational sector is given by the Einstein–Hilbert action with cosmological constant and with an $osp(2|8)$ -invariant background [33, 34]. In the case presented here, the extension to larger N is straightforward: the index i is allowed to run from 2 to $2s$, and the Lagrangian is a CS form for $osp(2s|8)$.

4.3. $D = 11$ Supergravity

In this case, the smallest adS superalgebra is $osp(32|1)$ and the connection is $\mathbf{A} = (1/2)\omega^{ab}J_{ab} + e^a J_a + (1/5!)b^{abcde}J_{abcde} + \bar{Q}\Psi$, where b is a totally antisymmetric fifth-rank Lorentz tensor one-form. Now, in terms of the elementary bosonic and fermionic fields, the CS form in (14) reads

$$L_{11}^{osp(32|1)}(\mathbf{A}) = L_{11}^{sp(32)}(\Omega) + L_F(\Omega, \Psi) \tag{33}$$

where

$$\Omega \equiv \frac{1}{2} \left(e^a \Gamma_a + \frac{1}{2} \omega^{ab} \Gamma_{ab} + \frac{1}{5!} b^{abcde} \Gamma_{abcde} \right)$$

is an $sp(32)$ connection. The bosonic part of (33) can be written as

$$L_{11}^{sp(32)}(\Omega) = 2^{-6} L_{G,11}^{adS}(\omega, e) - \frac{1}{2} L_{T,11}^{adS}(\omega, e) + L_{11}^b(b, \omega, e)$$

The fermionic Lagrangian is

$$\begin{aligned}
 L_F = & 6(\bar{\Psi}R^4D\psi) - 3[(D\bar{\Psi}D\psi) + (\bar{\Psi}R\psi)](\bar{\Psi}R^2D\psi) \\
 & - 3[(\bar{\Psi}R^3\psi) + (D\bar{\Psi}R^2D\psi)](\bar{\Psi}D\psi) \\
 & + 2[(D\bar{\Psi}D\psi)^2 + (\bar{\Psi}R\psi)^2 + (\bar{\Psi}R\psi)(D\bar{\Psi}D\psi)](\bar{\Psi}D\psi)
 \end{aligned}$$

where $R = d\Omega + \Omega^2$ is the $sp(32)$ curvature. The supersymmetry transformations (20) read

$$\begin{aligned}
 \delta e^a &= \frac{1}{8} \bar{\epsilon} \Gamma^a \psi, & \delta \omega^{ab} &= -\frac{1}{8} \bar{\epsilon} \Gamma^{ab} \psi \\
 \delta \psi &= D\epsilon, & \delta b^{abcde} &= \frac{1}{8} \bar{\epsilon} \Gamma^{abcde} \psi.
 \end{aligned}$$

Standard 11-dimensional supergravity [28] is an $N = 1$ supersymmetric extension of Einstein–Hilbert gravity that cannot accommodate a cosmological constant [35, 36]. An $N > 1$ extension of this theory is not known. In our case, the cosmological constant is necessarily nonzero by construction and the extension simply requires including an internal $so(N)$ gauge field coupled to the fermions, and the resulting Lagrangian is an $osp(32|N)$ CS form [25].

5. DISCUSSION

The supergravities presented here have two distinctive features: The fundamental field is always the connection \mathbf{A} and, in their simplest form, these are pure CS systems (matter couplings are discussed below). As a result, these theories possess a larger gravitational sector, including propagating spin connection. Contrary to what one could expect, the geometrical interpretation is quite clear, the field structure is simple, and, in contrast with the standard cases, the supersymmetry transformations close off shell without auxiliary fields.

The field content compares with that of the standard supergravities in $D = 5, 7, 11$ as shown in Table II.

Table II

D	Standard supergravity	CS supergravity
5	$e_\mu^a \Psi_\mu^\alpha \Psi_{\alpha\mu}$	$e_\mu^a \omega_\mu^{ab} \Psi_\mu^\alpha \Psi_{\alpha\mu} b$
7	$e_\mu^a A_{[3]} \Psi_\mu^{\alpha i} a_{ij}^i \lambda^\alpha \phi$	$e_\mu^a \omega_\mu^{ab} \Psi_\mu^{\alpha i} a_{ij}^i$
11	$e_\mu^a A_{[3]} \Psi_\mu^\alpha$	$e_\mu^a \omega_\mu^{ab} \Psi_\mu^\alpha b_{\mu}^{abcde}$

Some sector of these theories might be related to the standard supergravities if one identifies the totally antisymmetric part of the contorsion tensor in a coordinate basis, $k_{\mu\nu\lambda}$, with the Abelian 3-form, $A_{[3]}$. In 11 dimensions one could also identify the antisymmetric part of b with an Abelian 6-form $A_{[6]}$, whose exterior derivative $dA_{[6]}$, is the dual of $F_{[4]} = dA_{[3]}$. Hence, in $D = 11$ the CS theory possibly contains the standard supergravity as well as some kind of dual version of it [37, 38].

The field equations for these theories take the form

$$\langle \mathbf{F}^{n-1} G_A \rangle = 0 \quad (34)$$

where G_A are the generators of the superalgebra (see Appendix C). Spreading out these equations in terms of the Lorentz components (ω, e, b, a, ψ) produces somewhat involved expressions. It is therefore reassuring to verify that in all these theories the anti-de Sitter space is a fully SUSY background, and that for $\psi = b = a = 0$ there exist spherically symmetric, asymptotically adS standard black-hole solutions of the class discussed in ref. 23 as well as topological black holes of the type discussed in ref. 39. As will be discussed in a forthcoming paper, for the extreme cases, these solutions are also BPS states [40].

It would be natural to inquire about the stability or positivity of the energy for the linearized excitations around these or other solutions. This problem, however, is highly nontrivial. As shown in ref. 4, the phase space of a CS system splits up into separate regions where the symplectic form has different rank with radically different dynamical content. Thus, each CS action describes several distinct systems in fact. In a generic case (where the rank of the symplectic structure is maximal) the numbers of bosonic and fermionic degrees of freedom do not match—as it is also found in ref. 41—so the system cannot be naively related to a standard SUGRA [42].

It is possible to incorporate matter into these theories through a minimal coupling \mathbf{A} . This case has been recently discussed by Horava [43]. The matter currents must have equal-time commutators obeying the superalgebras above, which are typical for a system of extended objects. For $D = 11$, for example, the matter content is that of a theory with (super-) 0-, 2-, and 5-branes, whose respective worldhistories couple to the spin connection and the b fields.

Very recently a number of papers have appeared dealing with 11-dimensional CS theories and $OSp(32|11)$ [37, 38, 43–45]. In [43, 45], an interesting suggestion is given on how standard SUGRA could be obtained from a C-S theory of the type discussed here.

APPENDIX A. CHERN–SIMONS ACTION FOR $D = 2n - 1$

Consider the $(2n - 1)$ -dimensional CS Lagrangian defined by Eq. (14),

$$dL_{2n-1}^g = \langle \mathbf{F}^n \rangle \tag{35}$$

Integrating this equation, we can write L as [46, 47]

$$L = \frac{1}{(n + 1)!} \int_0^1 dt \langle \mathbf{A} \wedge (td\mathbf{A} + t^2\mathbf{A} \wedge \mathbf{A})^{n-1} \rangle + \alpha \tag{36}$$

where α is an arbitrary closed $(2n - 1)$ -form ($d\alpha = 0$).

Under a gauge transformation defined by a group element $g(x)$, the connection transforms as

$$\mathbf{A} \rightarrow \mathbf{A}^g = g^{-1}\mathbf{A}g + g^{-1}dg \tag{37}$$

The corresponding change in the Chern–Simons form is

$$L^g = L + d\beta + (-1)^{n-1} \frac{n!(n-1)!}{(2n-1)!} \langle (g^{-1}dg)^{2n-1} \rangle \tag{38}$$

where the $2(n - 1)$ -form β is a function of \mathbf{A} , and depends on g through the combination $g^{-1}dg$. Thus, the action

$$I_{CS} = \int_{\Sigma} L \tag{39}$$

which describes a gauge theory for the group \mathbf{G} , changes under a finite gauge transformation as the integral of (38), where the second term is a boundary term, and the third is proportional to the winding number.

One can see from (35) that under an infinitesimal gauge transformation (connected to the identity), the variation of L is a total derivative,

$$\delta_{\lambda}\mathbf{A} = \nabla\lambda \tag{40}$$

where λ is an arbitrary algebra-valued zero-form and $\nabla\lambda = d\lambda + [\mathbf{A}, \lambda]$. In fact, the R.H.S. of (35) is invariant under (40), e.g., δL is an exact form.

APPENDIX B. EXTENDED SUPERALGEBRAS

Let us look for the smallest supersymmetric extension of the anti-de Sitter algebra in dimension D . The strategy is as follows: The graded algebra will necessarily be of the form

$$\begin{aligned} [B, B] &\sim B \\ [B, F] &\sim F \end{aligned} \tag{41}$$

$$\{F, F\} \sim B$$

where the bosonic subalgebra is assumed to contain the generators of the adS group. The fermionic generators, on the other hand, must be in a spin-1/2 representation of the Lorentz group. The question now is, what is the minimal number of additional generators that is necessary to close the algebra? As was shown by van Holten and Van Proeyen [12], it is possible to construct the $N = 1$ supersymmetric extensions of adS for $D = 2, 3, 4 \pmod{8}$, but not for the remaining dimensions. Here we show that by relaxing the $N = 1$ condition it is possible to extend the result for the other cases.

In ref. 12, the result is found by imposing the Jacobi identity for the algebra (41). We will derive the same result and its extension by demanding that certain matrices constructed from Lorentz-invariant tensors form a representation of the algebra in a trivial way. This procedure excludes the exceptional supergroups from our analysis; however, there is only one case in which an exceptional supergroup is related to adS, that is, $F(4)$ for ($D = 6$) [29].

We will suppose that fermionic generators (F) are in the spin-1/2 representation of the Lorentz group,

$$F \sim Q_i^\alpha, \quad \alpha = 1, \dots, m; \quad i = 1, \dots, N \quad (42)$$

where $m = 2^{[D/2]}$, N is the number of supersymmetric generators, and i is an internal index unrelated to the Lorentz group.

In accordance with (41), we expect the generators of the adS group to be contained in the R.H.S. of the third anticommutator. A simple way to achieve this is by using the spinorial representation for the adS algebra

$$\begin{aligned} J_a &= \Gamma_a \\ J_{ab} &= \frac{1}{2} \Gamma_{ab} \end{aligned} \quad (43)$$

Additional constraints can be imposed on the spinors Q in order to produce smaller supersymmetric extensions of adS. There are only two restrictions compatible with Lorentz invariance (see, e.g., ref. 48): definite chirality (the Q 's are Weyl spinors), and "reality" (Majorana spinors). Chirality is only defined for even D . Here we shall not require chirality, but we demand the spinors to satisfy the following reality condition (modified Majorana spinors):

$$\bar{\psi}_i^\alpha = C^{\alpha\beta} u_{ij} \psi_j^\beta \quad (44)$$

which can be imposed for any D , provided the spacetime signature is $s - t = 0, 1, 2, 6, 7 \pmod{8}$ [48]. Here C and u are invertible matrices, the charge conjugation matrix and the invariant metric of the internal symmetry group,

respectively. Their inverses are defined with raised and lowered indices, respectively,

$$C_{\alpha\beta}C^{\beta\gamma} = \delta_{\alpha}^{\gamma} \tag{45}$$

$$u^{ij}u_{jk} = \delta_k^i$$

Without loss of generality, these metrics can be assumed to possess definite symmetry properties

$$C_{\gamma\beta} = \lambda C_{\beta\gamma}, \quad u^{lk} = \mu u^{kl} \tag{46}$$

with $\lambda, \mu = \pm 1$. The charge conjugation matrix is defined so that

$$(\Gamma^a)^T = \eta C \Gamma^a C^{-1} \quad \text{with } \eta^2 = 1 \tag{47}$$

An appropriate representation of the algebra (41) in which the fermionic generators satisfy the reality condition (44) is

$$Q_{\gamma}^k = \begin{bmatrix} 0 & \delta_{\gamma}^{\alpha} \delta_j^k \\ -C_{\gamma\beta} u^{ki} & 0 \end{bmatrix} \tag{48}$$

while the bosonic generators (B) are accommodated in the diagonal $m \times m$ and $N \times N$ blocks,

$$B \sim \begin{bmatrix} (G_{(k)})_{\beta}^{\alpha} & 0 \\ 0 & (M_{kl})_j^i \end{bmatrix} \tag{49}$$

The anticommutator of the fermionic generators (48) gives

$$\{Q_{\gamma}^k, Q_{\rho}^l\} = - \begin{bmatrix} u^{lk}(C_{\rho\beta} \delta_{\gamma}^{\alpha} + \mu C_{\gamma\beta} \delta_{\rho}^{\alpha}) & \\ & C_{\gamma\rho} (u^{ki} \delta_j^l + \lambda u^{li} \delta_j^k) \end{bmatrix} \tag{50}$$

The upper diagonal block on the R.H.S. can be expanded in a complete set of $m \times m$ matrices in the form

$$C_{\rho\beta} \delta_{\gamma}^{\alpha} + \mu C_{\gamma\beta} \delta_{\rho}^{\alpha} = A_{\gamma\rho}^{(k)} (\Gamma_{(k)})_{\beta}^{\alpha} \tag{51}$$

where $\Gamma_{(k)}$ stands for

$$\frac{1}{k!} \delta_{b_1 \dots b_k}^{a_1 \dots a_k} \Gamma^{b_1} \dots \Gamma^{b_k}, \quad 0 \leq k \leq D$$

and we have used the fact that the antisymmetric product of Γ 's is a basis of $M_{m \times m}$.

For each pair of indices (γ, ρ) the index α can be lowered in (51) multiplying by C . The result is a matrix with the same symmetry as u ,

$$[C\{Q_\gamma^k, Q_\rho^l\}]_{\alpha\beta} = \mu[C\{Q_\gamma^k, Q_\rho^l\}]_{\beta\alpha} \tag{52}$$

On the other hand, the symmetry of $C\Gamma^{(r)}$ depends on the sign of λ . In fact, from (47), it is easily seen that

$$(C\Gamma^a)^T = \eta\lambda C\Gamma^a \tag{53}$$

$$(C\Gamma^{ab})^T = -\lambda C\Gamma^{ab} \tag{54}$$

This implies that both $C\Gamma^a$ and $C\Gamma^{ab}$ have the same symmetry—opposite to that of C —only for $\eta = -1$. Furthermore, from this and (52), one concludes that the adS generators can occur on the R.H.S. of (50) if and only if

$$\lambda\mu = \eta = -1 \tag{55}$$

It can be shown that for each dimension there are always two possible representations for the charge conjugation matrix ($C_1 = \Gamma_1\Gamma_3 \dots, C_2 = \Gamma_2\Gamma_4 \dots$). It is easy to see that the signs of λ fixed for all dimensions *except* for $D = 2 \bmod 4$. For odd dimensions, $\eta = (-1)^{(D-1)/2}$, while for each even dimension $\eta = \pm 1$, with $\eta\lambda = (-1)^{[(D-2)/4]}$.

Thus, except for $D = 4k + 1$, C can always be chosen so that $\eta = -1$ and this uniquely fixes λ and μ . If $\mu = 1$, u is a symmetric quadratic form and hence the internal group is $SO(N)$. Conversely, for $\mu = -1$ the group is $Sp(N)$. Correspondingly, if $\lambda = -1$, the only matrices that enter on the R.H.S. of (51) are those for which $C\Gamma$ is symmetric, which is a basis for $sp(m)$. Conversely, if $\lambda = +1$, $C\Gamma$ is antisymmetric and the R.H.S. of (51) spans $so(m)$.

The semisimple algebras containing adS as a subalgebra in the bosonic sector would then be $osp(m|N)$ for $D = 2, 3, 4 \bmod 8$, and $osp(N|m)$ (with even N) for $D = 6, 7, 8 \bmod 8$. Their minimal extensions are $osp(m|1)$ and $osp(2|m)$, respectively.

The only exceptional case for which the representation (48) would not be appropriate occurs for $D = 5 \bmod 4$. In this case $\eta = +1$ and therefore the anticommutator of two Q 's of the form (48) would only contain either Γ^a or Γ_{ab} , but not both, and therefore it would not correspond to a supersymmetric extension of adS. In this case, the analysis is best carried out using complex Dirac spinors. One can repeat the previous construction, but taking the fermionic generators of the superalgebra in the complex representation

$$\bar{Q}_\gamma^l = \begin{bmatrix} 0 & \delta_\gamma^\alpha \delta_l^i \\ 0 & 0 \end{bmatrix} \tag{56}$$

$$Q_{\rho k} = \begin{bmatrix} 0 & 0 \\ -G_{\rho\beta} \delta_k^i & 0 \end{bmatrix} \tag{57}$$

where G is the Dirac conjugation matrix (for Minkowskian signature, $G = \Gamma^0$). Now the anticommutator of Q 's does not have a definite symmetry. In this case, the charge conjugation matrix can be chosen to be antihermitian and the upper block of $\{\bar{Q}, Q\}$ closes on the unitary algebra $u(m - 2, 2)$, while the lower block spans $u(N)$. Thus, for $D = 5 \pmod 4$, the superalgebra is $u(m - 2, 2|N)$, whose minimal extension is $su(m - 2, 2|1)$.

The previous discussion can be summarized in Table III.³

We indicate in brackets the alternative choices that satisfy $\lambda\mu = -1$, but $\eta = +1$. The corresponding supersymmetric theories do not have both the generator J^a in the bosonic sector and therefore the contact with gravity would be less transparent. Note that these ‘‘pseudo supergravities’’ exist for all even dimensions. Their algebras are different from those of the ‘‘real supergravities’’ for $D = 2 \pmod 4$, but are the same for $D = 4 \pmod 4$.

For $D \neq 5 \pmod 4$, the bosonic generators take the form

$$J_{(k)} = \begin{bmatrix} \frac{1}{2}(\Gamma_{(k)})^\alpha_\beta & 0 \\ 0 & 0 \end{bmatrix} \tag{58}$$

$$M^{kl} = \begin{bmatrix} 0 & 0 \\ 0 & (m^{kl})^i_j \end{bmatrix} \tag{59}$$

Table III

D	λ	η	μ	Superalgebras
2	-1 [+1]	-1 [+1]	+1 [-1]	$osp(2 N)$ [$osp(M 2)$]
3	-1	-1	+1	$osp(2 N)$
4	-1	-1 [+1]	+1	$osp(4 N)$
5	-1	+1	+1	$u(4 N)$
6	+1 [-1]	-1 [+1]	-1 [+1]	$osp(M 8)$ [$osp(8 N)$]
7	+1	-1	-1	$osp(M 8)$
8	+1	-1 [+1]	-1	$osp(M 16)$
9	+1	+1	-1	$u(16 N)$
10	-1 [+1]	-1 [+1]	+1 [-1]	$osp(32 N)$ [$osp(M 32)$]
11	-1	-1	+1	$osp(32 N)$
12	-1	-1 [+1]	+1	$osp(64 N)$
.
.
.

³ A special case occurs for $D = 3$, where we have defined the supergroup as $osp(2|1)$ only for completeness of the table. This group does not contain the generator of adS and, strictly speaking, one should have written $osp(2|1) \otimes sp(2)$ instead.

with $(m^{kl})_j^i = u^{ki}\delta_j^l + \lambda u^l i \delta_j^k$. The superalgebra now reads

$$\begin{aligned} [J^{(p)}, J^{(q)}] &= f_{(r)}^{(p)(q)} J^{(r)} \\ [J^{(p)}, M^{jl}] &= 0 \\ [J^{(p)}, Q_\alpha^l] &= \frac{1}{2} (\Gamma^{(p)})_\alpha^\beta Q_\beta^l \\ [M^{ij}, M^{kl}] &= f_{hm}^{ijkl} M^{hm} \\ [M^{jk}, Q_\alpha^l] &= (m^{jk})_i^l Q_\alpha^i \\ \{Q_\gamma^k, Q_\rho^l\} &= \frac{4}{m} u^{lk} (C\Gamma^{(p)})_{\gamma\rho} J_{(p)} - C_{\gamma\rho} M^{kl} \end{aligned} \quad (60)$$

APPENDIX C. EQUATIONS OF MOTION

Let Ω be a $2n$ -dimensional manifold with boundary $\partial\Omega = \Sigma$. Then, by Stokes' theorem the action can be written as

$$I_{cs} = \int_\Sigma L = \int_\Omega dL = \int_\Omega \langle \mathbf{F}^n \rangle \quad (61)$$

Varying I with respect to the connection, using $\delta\mathbf{F} = \nabla\delta\mathbf{A}$ and the Bianchi identity $\nabla\mathbf{F} = 0$, we obtain

$$\delta I_{cs} = n \int_\Omega \langle \nabla(\delta\mathbf{A}) \wedge \mathbf{F}^{n-1} \rangle = n \int_\Omega d\langle \delta\mathbf{A} \wedge \mathbf{F}^{n-1} \rangle \quad (62)$$

Using Stokes' theorem again, we obtain

$$\delta I_{cs} = n \int_\Sigma \delta\mathbf{A}^B \wedge \langle G_B \mathbf{F}^{n-1} \rangle \quad (63)$$

Thus, the equations of motion are

$$\langle \mathbf{F}^{n-1} G_B \rangle = 0 \quad (64)$$

These equations provide a representation of the supergravity algebra. Suppose the superalgebra has generators $G_A = \{B_a; Q_\alpha\}$, with a graded commutator algebra of the form

$$\begin{aligned} [B_a, B_b] &= C_{ab}^c B_c \\ [B_a, Q_\beta] &= C_{a\beta}^\gamma Q_\gamma \\ \{Q_\alpha, Q_\beta\} &= C_{\alpha\beta}^c B_c \end{aligned} \quad (65)$$

The connection associated to this algebra is $\mathbf{A} = \mathbf{W}^a B_a + \Psi^\alpha Q_\alpha = W + \Psi$. Calling D the covariant derivative restricted to the bosonic subgroup generated by B_a , by virtue of the Bianchi identity $\nabla \mathbf{F} = 0$, we obtain

$$D\mathbf{F} = [\mathbf{F}, \Psi] \tag{66}$$

Varying the action with respect to the connection $\delta \mathbf{A} = \delta W + \delta \Psi$ yields the field equations

$$\delta W: \langle \mathbf{F}^{n-1} B_a \rangle = 0 \tag{67}$$

$$\delta \Psi: \langle \mathbf{F}^{n-1} Q_\alpha \rangle = 0$$

Acting with D on the fermionic equation gives

$$D\langle \mathbf{F}^{n-1} Q_\alpha \rangle = \langle D(\mathbf{F}^{n-1}) Q_\alpha \rangle = \langle \mathbf{F}^{n-1} \{\Psi, Q_\alpha\} \rangle = 0 \tag{68}$$

where we have used (66) and the symmetry property of the bracket. Since $\{\Psi, Q_\beta\} = \Psi^\alpha \{Q_\alpha, Q_\beta\} = \Psi^\alpha C_{\alpha\beta}^c B_c$, the consistency condition is

$$\langle \mathbf{F}^{n-1} B_c \rangle C_{\alpha\beta}^c \Psi^\alpha = 0 \tag{69}$$

Supposing that Ψ^α is an arbitrary solution of the fermionic field equations, we conclude that

$$\langle \mathbf{F}^{n-1} B_c \rangle C_{\alpha\beta}^c = 0 \tag{70}$$

Thus, the consistency conditions associated with the fermionic equations do not produce additional restrictions on the theory. They are a subset of the bosonic field equations. If the structure constants $C_{\alpha\beta}^c$ were nonzero, the complete set of bosonic equations would enter in Eq. (70). The preceding analysis of the integrability conditions holds equally for Chern–Simons supergravities as for a standard supersymmetric theory.

APPENDIX D. CONTRACTION OF SUPER-ADS

In ref. 11 a family of CS theories was constructed for supersymmetric extensions of the Poincaré algebra of the form (7),

$$\{Q^\alpha, \bar{Q}_\beta\} = -i(\Gamma^a)_\beta^\alpha P_a - i(\Gamma^{abcde})_\beta^\alpha Z_{abcde} \tag{71}$$

It is possible to obtain this algebra by an appropriate contraction of the $N = 2$ superextension of an adS algebra, in the limit of vanishing cosmological constant. Let us define

therefore cannot be contained in \mathbf{M} .⁴ It is then reasonable to expect that, in analogy with what happens in three-dimensional gravity, the CS Poincaré supergravity theories of ref. 11 could be obtained from a contraction of the CS adS supergravities.

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⁴Note that, in this sense, $D = 11$ is an upper bound from a purely geometrical argument.

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